On the Early Termination of an Experiment

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1. Introduction

Many experiments consist of a series of independent observations on subjects or units that, for one reason or another, enter the experiment for treatment sporadically. Often a fixed sample size is agreed upon to make an inference or decision concerning the treatment. In a frequentist context, the agreed-upon sample size will generally depend on the power and size of the test of the hypothesis. In the Bayesian context, although a prespecified sample size is not a determining constituent for computing the posterior odds of one hypothesis versus another, planning for the costs and the administration of an experiment may lead to a determination of sample size prior to embarking on a trial. In any event where the experimental procedures are costly, it is of great interest for the investigator to know whether or not to continue testing a new treatment, drug or therapy after partially completing the experiment. We assume that the fixed size experiment requires N+M trials to reach a decision or conclusion and the investigator is interested in whether or not to continue the experiment until its agreed-upon prescribed termination, after observing N trials. We shall examine this problem from several viewpoints. Suppose it is agreed that N+M observations are to be taken, denoted by $X^{(N+M)} = (X^{(N)}, X_{(M)})$

$$X^{(N)} = (X_1, \ldots, X_N)$$
$$X_{(M)} = (X_{N+1}, \ldots, X_{N+M})$$

and a decision to be made depends on $T(X^{(N+M)})$. The latter could, of course, be $X^{(N+M)}$ itself. Assume $T(X^{(N+M)}) \in R$ and that a two decision problem is involved. Hence if $T(X^{(N+M)}) \in R_1$, $R_1 \cup R_1^C = R$ we decide d_1 otherwise we decide d_1^C or perhaps withhold a decision. We assume that $X^{(N)} = x^{(N)}$ has been observed and that we can calculate the predictive distribution function

$$F_{X_{(M)}}(x_{(M)} | x^{(N)}).$$

Hence, at least conceptually, it is clear that we can now calculate the conditional predictive distribution of T

$$F_T(t|x^{(N)})$$

and subsequently

$$P(T\in R_1|x^{(N)}) = P$$

assuming d_1 is the important decision, e.g. a new treatment is superior to a standard. Clearly if P is small, a continuation of the experiment is highly unlikely to lead to d_1 and we may decide to abandon the experiment. On the other hand if P is sufficiently large we are encouraged to continue the experiment. In other instances a prescribed sample size may yield an equivocal inference e.g. there appears to be some tendency for the new therapy to yield a better result than the standard but the posterior odds have not reached a required value. In such a case one may wish to calculate the probability that, if one continued to sample beyond the original prescription, the tendency would be established. Alternatively one could determine the probability of establishing the decision for each of various additional sample sizes. We shall illustrate these ideas of using predictive distributions for some standard situations.

2. Bernoulli Trials

Suppose we have a series of binary variates $X_1, X_2, \ldots$ that are independent conditional on $\theta = P(X_i=1) = 1-Pr(X_i=0)$ where $X_i=1$ is a successful outcome of therapy and $X_i=0$ is a failure.

Suppose a fixed sample size experiment of size N+M is used to test the hypothesis $\theta > a \geq 0$. Let us assume the following criterion; if

$S = \sum_{1}^{N+M} X_i \geq A$ the agent is to be declared effective. Suppose the experiment was performed sequentially and we had N observations already in hand and we wanted to decide whether it was worthwhile going on till the end of the trial noting that t out of N were successes. We could compute at this stage, for the given N and t, the predictive probability of S successes in N+M trials or equivalently R successes the next M trials

$$P = \Pr[S \geq A|N+M,t] = \Pr[t+R \geq A|M,t]$$

or since t is already known

$$P = \Pr[R \geq A-t|M,t].$$

We can calculate the above quantity, for example, if θ is assumed uniformly distributed a priori to be

$$P = \Pr[R \geq M-t] = \begin{cases} \sum_{r=A-t}^{M} \binom{t+r}{t}\binom{N+M-t-r}{N-t} \Big/ \binom{N+M+1}{N+1} & \text{if } A \geq t \\ 1 \text{ if } A \leq t \\ 0 \text{ if } N-t > N+M-A \end{cases}.$$

Now this is the probability that the goal will be reached by the end of the trial. If this is small enough, clearly there is not much point in continuing the trial. If the trial consists of a standard and a new treatment and for N_1+M_1 trials on the standard and N_2+M_2 on the new treatment there may be various values for (S_1,S_2) that indicate a new treatment superior to the standard, where S_1 and S_2 are the final number of successes if the trial were carried to completion. Since for any given pair (N_1,N_2) we have $(S_1=t_1+R_1, S_2=t_2+R_2)$, we could calculate the predictive distribution for R_1 and R_2 given (N_1,t_1) and (N_2,t_2) given priors for θ_1 and θ_2. We then suppose that the criterion for the new agent to be better than the standard will require that $(S_1,S_2) \in B_S$ which would imply $(R_1,R_2) \in B_R$ say. We would then calculate the probability of $(R_1,R_2) \in B_R$ to ascertain the worth of continuing the trial to its conclusion. Of course a determination to continue may depend not only on this probability but on several other factors as well. In particular, the region $(R_1,R_2) \in B_R$ will often arise from a decision involving whether $(\theta_1,\theta_2) \in B_\theta$. In the next few sections we shall apply the procedure to simple tests of whether a parameter lies in some interval.

3. Poisson Sampling

Suppose a random sample from a Poisson distribution is to be taken where

$$f(x) = \frac{e^{-\lambda}\lambda^{x}}{x!} \qquad x = 0,1,\ldots$$

For a test H_0: $\lambda > a$ versus H_1: $\lambda \le a$, we decide that $\lambda \le a$ if and only if

$$P(\lambda \le a | x^{(N+M)}) \ge p.$$

If we assume that $p(\lambda) \propto \lambda^{-1}$ then for

$$y = \sum_{i=1}^{N+M} x_i = y_N + y_M \qquad y_N = \sum_{1}^{N} x_i, \quad y_M = \sum_{M+1}^{N+N} x_i$$

$$p(\lambda|y) = \frac{\lambda^{y-1}(N+M)^{y} e^{-(N+M)\lambda}}{(y-1)!}, \ \lambda > 0, \ y > 0.$$

Hence $2(N+M)\lambda$ is a χ^2 variate with 2y degrees of freedom. Hence

$$P[\lambda \le a|y] = F_{2y}(2(N+M)a) \ge p$$

with F_{2y} the distribution function of a χ^2 with 2y degrees of freedom. Hence

$$2(N+M)a \le F_{2y}^{-1}(p).$$

Then after observing y_N we need to calculate

$$P = P[F^{-1}_{2y_N+2Y_M}(p) \geq 2(N+M)a]$$

using the distribution of $Y_M = \sum_{N+1}^{N+M} X_i$ conditional on y_N.

Now the predictive probability function of Y_M given y_N is easily obtained as a negative binomial where

$$P[Y_M = t|y_N] = \binom{y_N+t-1}{t} \left(\frac{M}{M+N}\right)^t \left(\frac{N}{M+N}\right)^{y_N}$$

for $t = 0,1,\ldots$

Further, the actual computation

$$P = P[F^{-1}_{2y_N+2Y_M}(p) \geq 2(N+M)a]$$

is more easily accomplished by finding the largest value of y which satisfies

$$F_{2y}(2(N+M)a) \geq p$$

say y_a. Then we calculate

$$P = \begin{cases} \sum_{t=0}^{y_a-y_N} \binom{y_N+t-1}{t} \left(\frac{M}{M+N}\right)^t \left(\frac{N}{M+N}\right)^{y_N} & \text{if } y_a \geq y_N > 0 \\ 0 & \text{if } y_a < y_N \end{cases}$$

In an experiment intended to determine whether the mean number of red blood corpuscles per cell is greater than 1, a sample of 20 cells yielded the following results:

No. corpuscles	0	1	2	3	4
No. cells	7	8	3	1	1.

We assume that $p(\lambda) \propto \lambda^{-1}$. Hence given the data $y = 21$, $a = 1$, $N = 20$

$$P[\lambda > 1|21] = 1 - F_{42}(40) = .56.$$

Suppose we were to sample another 5 cells and inquire as to whether the predictive probability that $P[\lambda > 1]$ will be least as large as various values of p. The computations are given using the previous work:

p	.900	.863	.818	.763	.708	.629	.553
P	.017	.032	.061	.108	.182	.287	.577

Hence an additional sample of 5 is very unlikely to increase the posterior probability $P(\lambda > 1)$ appreciably.

4. Exponential Sampling

Suppose now $X_1,\ldots,X_N,X_{N+1},\ldots,X_{N+M}$ are independently and identically distributed as

$$f(x) = \theta e^{-\theta x}.$$

Assume that we wish to test H_0: $\theta \le a$ versus H_1: $\theta > a$. We assume a prior density $p(\theta)$ for θ, and decide that $\theta \le a$ if

$$P(\theta \le a|x^{(N+M)}) \ge p.$$

Suppose that we assume that $p(\theta) \propto \theta^{-1}$ so that the posterior distribution of $2\theta(N+M)\bar{x}_{N+M}$ is a χ^2_{2N+2M} variate. Hence we require that

$$P(\theta \le a|x^{(N+M)}) = F(2a(N+M)\bar{x}_{N+M}) \ge p$$

where $F(\cdot)$ is the distribution function of a χ^2_{2N+2M} variate. Further

$$2a(N+M)\bar{x}_{N+M} \geq F^{-1}(p)$$

$$M\bar{x}_M \geq \frac{1}{2a} F^{-1}(p) - N\bar{x}_N.$$

Now if we stopped after the first N observations we can calculate the predictive probability

$$P = P[M\bar{X}_M \geq \frac{1}{2a} F^{-1}(p) - N\bar{x}_N].$$

It is easy to show that the predictive distribution of

$$\bar{X}_M \sim \bar{x}_N Y$$

where Y is an F-variate with 2M and 2N degrees of freedom. Hence

$$P = P\left(\frac{\bar{X}_M}{\bar{x}_N} \geq \frac{F^{-1}(p)}{2aM\bar{x}_N} - \frac{N}{M}\right)$$

$$= 1 - F_{2M,2N}\left(\frac{1}{2aM\bar{x}_N} F^{-1}(p) - \frac{N}{M}\right). \qquad (4.1)$$

where $F_{2M,2N}(y)$ is the distribution function of Y.

We note that if we assume a conjugate prior density

$$p(\theta) \propto \theta^{K-1} e^{-Kx_0\theta}$$

then we would obtain

$$P(\theta \geq a | x^{(N+M)} = F(2a(N\bar{x}_N+M\bar{x}_M+Kx_0)) \leq p$$

where F is the distribution function of a χ^2 with 2(M+N+K) degrees of freedom.

Now

$$2a(N\bar{x}_N+M\bar{x}_M+Kx_0) \le F^{-1}(p)$$

and

$$P = 1 - F_{2M,2(N+K)}\left(\frac{F^{-1}(p)(N+K)}{2aM(N\bar{x}_N+Kx_0)} - \frac{N+K}{M}\right).$$

As an example we use the data in Shapiro and Whitney (1967,p.535). The length of life in hours of 10 projector lamps were obtained as 2, 18, 6, 3, 8, 26, 5, 11, 15, 5. It is assumed that the data are a random sample from an exponential distribution. To test that $\theta < \frac{1}{9}$ i.e. the mean life of these projector lamps exceeds 9 hours, using $p(\theta) \propto \theta^{-1}$ we calculate

$$P(\theta < \tfrac{1}{9}|x^{(10)}) = F_{20}\left(\frac{2\times10\times9.9}{9}\right) = F_{20}(22) = .659,$$

where $F_n(\cdot)$ is the distribution function of a χ^2 variate with n degrees of freedom. If we are only willing to assert that $\theta < \frac{1}{9}$ if the posterior probability of that event is not less than .7, we may ask for the chance of achieving this if we take a further sample of 10 observations. Now we require

$$P(\theta < \tfrac{1}{9}|x^{(20)}) = F_{40}\left(\frac{2\times20\times\bar{x}_{N+M}}{9}\right) \ge .7$$

so that using (4.1) we obtain

$$P = 1 - F_{20,20}(.988) = .51.$$

Hence the odds are about even that the goal would be achieved with a further sample of 10 observations.

5. Sampling From a Normal Distribution with $\sigma^2 = 1$

Suppose $X_1,\ldots,X_N,X_{N+1},\ldots,X_{N+M}$ are independently and identically distributed as $N(\mu,1)$. Suppose that we wish to test H_0: $\mu \le a$ vs. H_1: $\mu > a$. Assume a prior for μ, $p(\mu)$ that leads to

$$p(\mu|x^{(N+M)}).$$

We then suppose that we will decide that $\mu \le a$ if and only if

$$P(\mu \le a|x^{(N+M)}) \ge p$$

for some arbitrarily specified p.

Now for the sake of an example assume that $p(\mu) \propto$ const., so that the posterior distribution of μ is $N\left(\bar{x}_{N+M},\frac{1}{N+M}\right)$. Then

$$P(\mu \le a|x^{(N+M)}) = \Phi\left(\frac{a-\bar{x}_{N+M}}{\sqrt{\frac{1}{N+M}}}\right) \ge p$$

or

$$\sqrt{N+M}(a-\bar{x}_{N+M}) \ge \Phi^{-1}(p)$$

or

$$(N+M)a - \Phi^{-1}(p)\sqrt{N+M} - N\bar{x}_N \ge M\bar{x}_M.$$

Now if we have already observed $X_1,\ldots,X_N$ then we compute for fixed $\bar{x}_N$ and future $\bar{X}_M$

$$P = \Pr\left(\bar{X}_M \le \frac{a(N+M)}{M} - \frac{(N+M)^{1/2}\Phi^{-1}(p)}{M} - \frac{N}{M}\bar{x}_N\right).$$

Since the predictive distribution of $\bar{X}_M$ is $N\left(\bar{x}_N, \frac{1}{M}+\frac{1}{N}\right)$ then

$$P = \Phi\left\{\left(\frac{N}{M}\right)^{1/2}[(a-\bar{x}_N)(N+M)^{1/2} - \Phi^{-1}(p)]\right\}.$$

Clearly for fixed values a, N, M, and p, P increases monotonically with decreasing $\bar{x}_N$.

6. Normal Sampling with μ and σ^2 Unknown

Let X_i, $i = 1,\ldots,N+M$ be independently and identically distributed as $N(\mu,\sigma^2)$. For a test of $\sigma^2 \le a$ versus $\sigma^2 > a$ we decide that $\sigma^2 \ge a$ if

$$P(\sigma^2 \ge a | x^{(N+M)}) \ge p.$$

In particular, let

$$p(\mu,\sigma^2) \propto \frac{1}{\sigma^2} .$$

Then the posterior distribution of

$$\frac{\nu s^2_{N+M}}{\sigma^2} \text{ is } \chi^2_\nu \text{ for } \nu = N+M-1, \text{ where}$$

$$\nu s^2_{N+M} = \sum_1^{N+M} (x_i - \bar{x}_{N+M})^2 \quad \text{and} \quad (N+M)\bar{x}_{N+M} = \sum_1^{N+M} x_i .$$

Hence

$$P\left(\sigma^2 \ge a | x^{(N+M)}\right) = P\left(\frac{\nu s^2_{N+M}}{\sigma^2} \le \frac{\nu s^2_{N+M}}{a^2}\right)$$

$$= F_\nu\left(\frac{\nu s^2_{N+M}}{a}\right)$$

where F_ν is the distribution function of a χ^2_ν variate. Now for

$$F_\nu\left(\frac{\nu s_{N+M}^2}{a}\right) \geq p$$

$$\frac{\nu s_{N+M}^2}{a} \geq F_\nu^{-1}(p).$$

Since

$$\nu s_{N+M}^2 = (N-1)s_N^2 + (M-1)s_M^2 + \frac{NM}{N+M}(\bar{x}_M - \bar{x}_N)^2, \text{ where}$$

$$(N-1)s_N^2 = \sum_1^N (x_i - \bar{x}_N)^2,\ N\bar{x}_N = \sum_1^N x_i,\ (M-1)s_M^2 = \sum_{N+1}^{N+M}(x_i - \bar{x}_M)^2$$

and

$$M\bar{x}_M = \sum_{N+1}^{N+M} x_i$$

we can, after observing $x^{(N)}$, calculate

$$\begin{aligned} P &= P\left[(N-1)s_N^2 + Y + \frac{NM}{N+M}(\bar{X}_M - \bar{x}_N)^2 \geq aF_\nu^{-1}(p)\right] \\ &= P\left[Y + \frac{NM}{N+M}(\bar{X}_N - x_N)^2 \geq aF_\nu(p) - (N-1)s_N^2\right] \end{aligned}$$

where Y is the random variable representing the unobserved $(M-1)s_M^2$. Hence in order to evaluate the above we must calculate the joint predictive distribution of Y and $\bar{X}_M$ given $x^{(N)}$. First we note for N > 1,

$$p(\mu,\sigma^2|x^{(N)}) = \frac{z^{\frac{N-1}{2}} e^{-\frac{z}{2\sigma^2}} \sqrt{N}\, e^{-\frac{N}{2\sigma^2}(\bar{x}_N-\mu)^2}}{2^{\frac{(N-1)}{2}} \sigma^{N+1}\, \Gamma\left(\frac{N-1}{2}\right) \sigma\sqrt{2\pi}}$$

where $z = (N-1)s_N^2$. Setting $\bar{X}_M = X$, the predictive density of X and Y is

$$f(x,y|x^{(N)}) = \int f(x,y|\mu,\sigma^2)p(\mu,\sigma^2|x^{(N)})d\mu d\sigma^2$$

where for M > 1

$$f(x,y|\mu,\sigma\ ) = \frac{\sqrt{M}\ e^{-\frac{M}{2\sigma^2}(x\ -\ \mu)^2}\ y^{\frac{M-1}{2}-1}\ e^{-\frac{z}{2\sigma^2}}}{\sigma\sqrt{2\pi}\ 2^{\frac{(M-1)}{2}}\ \Gamma\left(\frac{M-1}{2}\right)\ \sigma^{M-1}}.$$

Since

$$M(x-\mu)^2 + N(\bar{x}_N - \mu)^2 = (N+M)(\mu-w)^2 + \frac{NM}{N+M}(x - \bar{x}_N)^2$$

where $(N+M)w = N\bar{x}_N + Mx$, integration with respect to μ and σ yield for M > 1

$$f(x,y|x^{(N)}) = \frac{\sqrt{MN}\ \Gamma\left(\frac{M+N-1}{2}\right)\ z^{\frac{N-1}{2}}\ y^{\frac{M-3}{2}}}{\sqrt{M+N}\ \Gamma\left(\frac{1}{2}\right)\ \Gamma\left(\frac{N-1}{2}\right)\ \Gamma\left(\frac{M-1}{2}\right)}\left[z + y + \frac{NM}{N+M}(x-\bar{x}_N)^2\right]^{-\frac{M+N-1}{2}}.$$

Now we need to calculate

$$P = P\left[\frac{Y + \frac{NM}{N+M}(X - \bar{x}_N)^2}{z} \geq \frac{aF_\nu^{-1}(p)}{z} - 1\right]$$

or

$$P = P\left[\frac{(N-1)\left[Y + \frac{NM}{N+M}(X - \bar{x}_N)^2\right]}{Mz} \geq \frac{a(N-1)F_\nu^{-1}(p)}{Mz} - \frac{N-1}{M}\right].$$

From the joint predictive density of X and Y we can easily show that

$$\frac{(N-1)\left[\ Y + \frac{NM}{N+M}(X - \bar{x}_N)^2\ \right]}{Mz}$$

is an F-variate with M and N-1 degrees of freedom. Hence

$$P = 1 - F_{M,N-1}\left(\frac{aF_{\nu}^{-1}(p)}{Ms_N^2} - \frac{N-1}{M}\right).$$

It can easily be shown that the above holds for M = 1 as well.

Under the same circumstances as in the previous test for σ^2 suppose we wish to test $\mu \leq a$ versus $\mu > a$. Now the posterior density of

$$\frac{(\mu - \bar{x}_{N+M})\sqrt{N+M}}{s_{N+M}}$$

is t with $\nu = N+M-1$ degrees of freedom. Hence we decide on $\mu \leq a$ if

$$P[\mu \leq a | x^{(N+M)}] = S_{\nu}\left[\frac{(a - x_{N+M})\sqrt{N+M}}{s_{N+M}}\right] \geq p,$$

where $S_{\nu}(\cdot)$ is the distribution function of t with ν degrees of freedom. Hence

$$\frac{(a-\bar{x}_{N+M})\sqrt{N+M}}{s_{N+M}} \geq S_{\nu}^{-1}(p).$$

After observing $x^{(N)}$ we need to calculate

$$P\left(\frac{\left[a - \frac{N\bar{x}_N}{N+M} - \frac{MX}{N+M}\right](N+M)^{1/2}(N+M-1)^{1/2}}{\left[z + Y + \frac{NM}{N+M}(X - \bar{x}_N)^2\right]^{1/2}} \geq S_{\nu}^{-1}(p)\right) = P.$$

The random variable on the left within the bracket is a function of X and Y whose joint density was previously given. However the requisite random variable is not of a standard form and its exact distribution would require extensive tabling.

As an approximation for P when N is sufficiently large let

$$S_\nu\left[\frac{(a - \bar{x}_{N+M})\sqrt{N+M}}{s_N}\right] \doteq S_\nu\left[\frac{(a - \bar{x}_{N+M})\sqrt{N+M}}{s_{N+M}}\right] \geq p.$$

Then calculate

$$P \doteq P\left[\frac{(a - \bar{x}_{N+M})\sqrt{N+M}}{s_N} \geq S_\nu^{-1}(p)\right]$$

$$= P\left[\frac{(a - \bar{x}_N) + (\bar{x}_N - X)\frac{M}{N+M}}{s_N} \geq S_\nu^{-1}(p)\right]$$

$$= P\left[\frac{(\bar{x}_N - X)\sqrt{MN}}{s_N\sqrt{N+M}} \geq \left(\frac{N}{M}\right)^{1/2}(N+M)^{1/2}\left(S_\nu^{-1}(p) + \frac{\bar{x}_N - a}{s_N}\right)\right]$$

or

$$P \doteq 1 - S_{N-1}\left[\left(\frac{N}{M}\right)^{1/2}(N+M)^{1/2}\left(S_\nu^{-1}(p) + \frac{\bar{x}_N - a}{s_N}\right)\right].$$

This result may easily be extended to the case of a conjugate prior distribution for μ and σ^2. It essentially entails a modification of $\bar{x}_N$, s_N, s_{N+M} and the degrees of freedom of the student distribution in the formula for P.

7. Other Applications

Consider an analagous situation in terms of a frequentist significance test where a fixed sample size of N+M observations is to be taken. Further assume that the rejection of the null hypothesis is at level α. Rejection of H_0 then requires that

$$P[T(X^{(N)}, X_{(M)}) \in R] \le \alpha.$$

Now a Bayesian statistician could calculate, after N observations were in hand,

$$P = P[T(x^{(N)}, X_{(M)}) \in R | x^{(N)}]$$

and advise a frequentist on the possibility of achieving significance if the experiment were continued for another M observations. Presumably a "non-informative" prior would be used to make the Bayesian approach conform as much as possible to frequentist procedures. Of course a flexible frequentist could make the calculation and use it as an index of potential rejection at level α without necessarily conceiving of it as a relative frequency.

We shall give as an example a significance test for a random sample from the exponential distribution.

Here H_0: $\theta = a$ versus H_1: $\theta < a$ and the test will reject H_0 if

$$2(N+M)\bar{x}_{N+M} \ge \frac{y_\alpha}{a}$$

where

$$P[2a(N+M)\bar{X}_{N+M} \ge y_\alpha | H_0] \le \alpha$$

where the random variable to the left of the inequality sign above is distributed as a χ^2_{2N+2M} variate. Now assuming the previously used non-informative prior we calculate the predictive probability of rejection

$$P^* = P[2a(N\bar{x}_N + M\bar{X}_M) \ge y_\alpha | \bar{x}_N, H_0]$$

$$= P\left[\frac{M\bar{X}_M}{N\bar{x}_N} \geq \frac{y_\alpha}{2aN\bar{x}_N} - 1\right]$$

$$= P\left[\frac{\bar{X}_M}{\bar{x}_N} \geq \frac{y_\alpha}{2aM\bar{x}_N} - \frac{N}{M}\right]$$

$$= 1 - F_{2M,2N}\left(\frac{y_\alpha}{2aM\bar{x}_N} - \frac{N}{M}\right).$$

A sufficiently small value for P^* would provide some guidance on the early termination for the experiment.

In the exponential example of section 4 if we set the significance level at $\alpha = .3$ then a test at that level for $N = 10$ would not reject $\theta = \frac{1}{9}$ since

$$\frac{2N\bar{x}}{9} = 22 \leq 22.8 = y_{.3,20}$$

where $y_{\alpha,n}$ is the percentage point of a χ_n^2 distribution. Now if we were to double the size of the sample, as previously, by setting $M = 10$ then $y_{.3,40} = 44.2$ and

$$P^* = 1 - F_{20,20}\left(\frac{44.2\times 9}{2\times 10\times 9.9} - 1\right) = 1 - F_{20,20}(1.009) = .492.$$

Hence, even when doubling the sample size the chance is still less than .5 that significance would be attained at $\alpha = .3$ when it originally was at

$$P[\chi_{20}^2 \geq 22] = .36.$$

8. Remarks

While we have considered in this paper decisions about parameters from conditionally independent copies, clearly many problems would involve nonidentically-distributed variables and the continuation of sampling could also involve questions about optimal design of the future observations that could more speedily lead to a decision or inference.

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Reference

Shapiro, I.M. and Whitney, D.R. (1967) Elementary Analysis and Statistics. Merrill, Columbus.

ERRATA

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Page	Line	Original	Corrected
3	8	$\Pr[R \geq M-t]$	$\Pr[R \geq A-t \mid M,t]$
4	7	$Y_M = \sum_{M+1}^{N} x_i$	$Y_M = \sum_{N+1}^{N+M} x_i$
7	2 from bottom	$P(\theta \geq a \mid x^{(N+M)}) \leq p$	$P(\theta \leq a \mid x^{(N+M)}) \geq p$
12	3	$e^{-\frac{z}{2\sigma^2}}$	$e^{-\frac{y}{2\sigma^2}}$
14	5	$S_\nu^{-1}(p)$	$\frac{S_\nu^{-1}(p)}{\sqrt{N+M}}$
14	6	$(N+M)^{1/2}[S_\nu^{-1}(p) + \frac{\bar{x}_N - a}{s_N}]$	$S_\nu^{-1}(p) + \frac{(N+M)^{1/2}(\bar{x}_N - a)}{s_N}$
14	8	Corrected formula given as	

$$P \doteq 1 - S_{N-1}\left[\left(\frac{N}{M}\right)^{1/2}\left(S_\nu^{-1}(p) + \frac{(N+M)^{1/2}(\bar{x}_N - a)}{s_N}\right)\right]$$